Donaldson = Seiberg-Witten

from Mochizuki's formula and instanton counting

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X : C^∞ 4-mfd $b^+>1$, $b_1=0$

$(K_X^2) := 2\chi(X)+3\sigma(X)$ $\chi_h(X) := \dfrac{\chi(X)+\sigma(X)}{4}$ as if X : cpx surface

▷ Donaldson invariants

$$\mathcal{D}^{\xi}(\exp(\alpha z+px)) := \sum_{y} \Lambda^{\dim M(y)} \int_{M^U(y)} \exp(\mu(\alpha z+px))$$

z, x : variable $\alpha \in H_2(X)$, $p = pt \in H_0(X)$

$y=(2,\xi,n) \in H^*(X)$ ξ : fix, but move n

$M(y)$: moduli of $U(2)$-instanton with Chern class $= y$

$M^U(y)$: Uhlenbeck compactification

$= M(y) \sqcup X\times M(y-(0,0,1)) \sqcup S^2X \times M(y-(0,0,2)) \sqcup \cdots$

$\mathcal{E}$: universal b'dle on $X\times M(y)$

$$\mu(\alpha z+px) = \int_X \left(c_2(\mathcal{E})-\tfrac{1}{4}c_1(\mathcal{E})^2\right)\cup(\alpha z+px)$$

<u>Def.</u> X : KM-simple type $\overset{\text{def.}}{\Longleftrightarrow} \left(\dfrac{\partial^2}{\partial x^2}-4\Lambda^2\right)\mathcal{D}^{\xi}=0$

▷ Seiberg-Witten invariants

$\mathfrak{s}$: spin^c str. $\quad c_1(\mathfrak{s}) = c_1(S^+) \rightsquigarrow SW(\mathfrak{s}) \in \mathbb{Z}$

$\begin{cases} \phi : \text{spinor} \\ A : \text{spin connection} \end{cases} \qquad \begin{cases} \not{D}_A \phi = 0 \\ F_A^+ = \mu(\phi) \end{cases}$

Def. X : SW simple type $\iff$ SW inv. $\neq 0$ only if v.dim. of moduli sp. $= 0$

$SW(\mathfrak{s})$: SW invariant $\underset{\text{def.}}{\iff} c_1(\mathfrak{s})^2 = (K_X^2)$

No non-simple type 4-mfd is found so far.

Witten's conjecture (1994)

X : SW simple type

$\implies$ KM simple type &

$$D^3(\exp \alpha \times (1 + \tfrac{1}{2}p)) = 2^{(K_X^2) - \chi_h(X) + 2} (-1)^{\chi_h(X)}$$

$$\times\, e^{(\alpha^2)/2} \sum_{\mathfrak{s}} SW(\mathfrak{s}) (-1)^{(\xi, \xi + c_1(\mathfrak{s}))/2} e^{(c_1(\mathfrak{s}), \alpha)} \quad \text{(finite sum)}$$

Witten's idea

Write Donaldson invariants by path integral

······ Euler class of ∞-rk vector bdle / ∞-dim mfd

$(\infty - \infty = \text{finite})$

change g by tg $(t>0)$

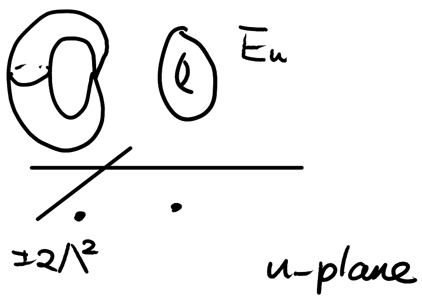
small t ···· Euler class localises to the zero set of a natural section $=$ instanton moduli sp.

large t ···· described by "vacuum states" on $\mathbb{R}^4$

On $\mathbb{R}^4$

SW curves : $y^2 = 4x(x^2+ux+\Lambda^4)$

control the gauge theory !



a family of elliptic curves

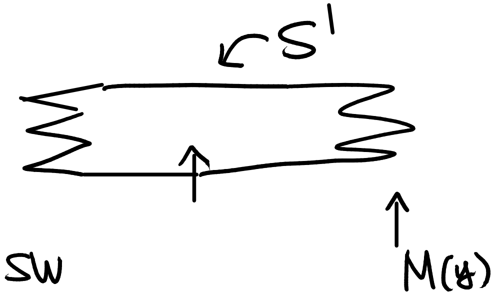
The contribution only comes from $u = \pm 2\Lambda^2$ ···· elliptic curve is singular

$\Longrightarrow$ SW invariants

mathematical approach
- Pidstrigach – Tyurin
- Feehan – Leness

SO(3)-monopole moduli
cobordism

with a fund. matter



$\Rightarrow$ Witten's conj. when $(K_X^2) \geq \chi_h(X) - 3$
or abundant

computation of local contribution around fixed pts
difficult because of singularity

- Mochizuki : X : cpx proj. surface
  - Use algebro-geometric model of SO(3)-monopole moduli
  - Develop perfect obstruction theory

explicit formula of local contribution
in terms of integration over Hilb. scheme of pts on X

GNY : Computation of the integral.

1°. enough to compute for X : toric surface

2°. localization $\Longrightarrow$ Nekrasov partition func. $X = \mathbb{R}^4$

$\approx$ equivariant Donaldson invariants

+ 3°. Nekrasov partition func. can be computed via SW curves

Th. 1) Suppose X : cpx projective

$$\Longrightarrow \mathcal{D}^{\mathfrak{s}}(\exp(\alpha z + p x)) = \sum_{\mathfrak{s}} SW(\mathfrak{s}) \operatorname*{Res}_{a=\infty} \mathcal{B}(\mathfrak{s}, \mathfrak{z}; a) da$$

$\mathcal{B}(\mathfrak{s}, \mathfrak{z}; a)$: differential explicitly written in terms of Nekrasov partition function with a single fund. matter

(Rem. $u \approx a^2$ in SW curve)

This formula makes sense even for X : C^∞ 4-mfd.

Conjecture 1) is true also for C^∞ 4-mfd X .

Assume conjecture:

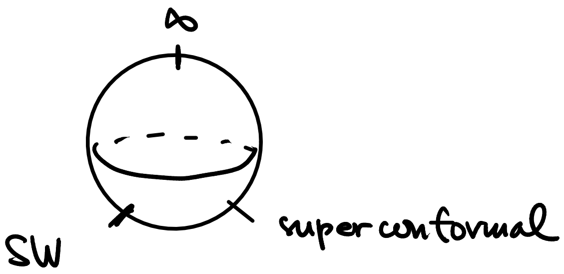
2) $\mathcal{B}(\mathfrak{s}, \mathfrak{z} : a) da$; (after change of variable $a \to \phi^{(4)}$)
extends to a meromorphic
differential defined over $\mathbb{P}^1$

3) It has 3 possible poles

a) $\phi^4 = \infty$, b) SW contribution

and c) superconformal point

[Mariño-Moore-Peradze]



Residue thm $\Longrightarrow$ $\underset{\phi^4=\infty}{\mathrm{Res}} + \underset{SW}{\mathrm{Res}} + \underset{s.c.\,pt}{\mathrm{Res}} = 0$

$\Downarrow$ 1) $\Downarrow$ Witten's conjecture $\Downarrow$??

Ref (Mariño - Moore - Peradze)

Assume X: SW simple type $\quad X$: superconformal simple type

$\Longleftrightarrow$ a) $(K_X^2) \geq \chi_h(X) - 3$

def. or b) $\displaystyle\sum_{\mathfrak{s}} (-1)^{(K_X, K_X + c_1(\mathfrak{s}))/2} SW(\mathfrak{s}) (c_1(\mathfrak{s}), \alpha)^n = 0$

$$0 \leqq {}^{\forall} n \leqq \chi_h(X) - (K_X^2) - 4$$

e.g. X: elliptic surface

(Thm cont'd)

obviously true

4) Donaldson inv. depends only on $\mathfrak{z}$ mod 2 (up to sign)

$\Longrightarrow$ X : superconformal simple type

$\Longrightarrow \displaystyle\sum_{\mathfrak{s}} SW(\mathfrak{s}) \mathcal{B}(\mathfrak{s}, \mathfrak{z} : a) da$ is regular at superconformal pt.

5) Residue Thm $\Longrightarrow$ Witten's conjecture is true.

<u>Rem.</u> ① If one can develop "perfect obstruction theory" on the Uhlenbeck cptification, it should give the same formula for C^∞ 4-mfd.

or maybe need to introduce "Gieseker cptification" for almost cpx surfaces.

② extension of $\mathcal{B}(\frac{t}{?}, 3; a)\,da$ is nontrivial.

Φ^4 is defined as a generating func. of $\int_{M(u)}$

$\Rightarrow$ Φ^4 is defined only at the formal n.b.d. of ∞.

interior pts of S^2 are not geometric.

Now I explain Nekrasov partition function

○ partition function for $N=2$ $SU(2)$ SUSY YM
with a single fund. matter (after Nekrasov)

$M(n) = M(2,n)$: framed moduli space of torsion-free sheaves on $\mathbb{P}^2$

$= \{ (E, \varphi) \mid E$: rk 2 torsion-free sheaf on $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$
$c_2(E) = n$ $\varphi : E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}^{\oplus 2} \} / \text{isom}$

<u>Fact</u>. ① $M(n)$: smooth of $\dim_{\mathbb{C}} = 4n$

② $\pi : M(n) \longrightarrow M^U(n)$
Uhlenbeck (partial) compactification
resolution of singularities

$M(n) = M(2,n) \curvearrowleft T^3 = T^2 \times \mathbb{C}^*$

$\left(\begin{array}{l} \mathbb{C}^* \subset SL(2,\mathbb{C}) \text{ acts by the change of framings } \varphi \\ T^2 \curvearrowright \mathbb{C}^2 \quad (x,y) \mapsto (t_1 x, t_2 y) \end{array} \right.$

$\mathrm{Lie}\, T^3 = \mathbb{C}\varepsilon_1 \oplus \mathbb{C}\varepsilon_2 \oplus \mathbb{C}a$

matter bdle $\quad \mathcal{V}_{(E,\varphi)} = H^1(E(-\ell_\infty)) \otimes \underbrace{K_{\mathbb{C}^2}^{1/2}}_{\cong\, e^{-\varepsilon_1+\varepsilon_2/2}}$

$\curvearrowleft S^1$ multiplication

$\mathrm{Lie}\, S^1 = \mathbb{C}m \qquad$ (matter)

$$Z^{\mathrm{in}}(\varepsilon_1, \varepsilon_2, a, m, \Lambda) := \sum_n \Lambda^{3n} \int_{M(n)} e(\mathcal{V} \otimes e^m)$$

$$\in \mathbb{Q}(\varepsilon_1, \varepsilon_2, a, m)[[\Lambda]]$$

— definition by localization

fixed pts ··· pair of Young diagram

$$= \sum_{\vec{Y}=(Y^1,Y^2)} \frac{e(H^1(I_{Y_1}(-\ell_\infty)) \otimes e^m)\, e(H^1(I_{Y_2}(-\ell_\infty)) \otimes e^m)}{\prod_{\alpha,\beta=1,2} e(\mathrm{Ext}^1(I_{Y_\alpha}, I_{Y_\beta}(-\ell_\infty)) \otimes e^{a_\beta - a_\alpha})}$$

$$(a_2 = a,\ a_1 = -a)$$

<u>Rem</u> • pure theory : replace $e(\mathcal{V}\otimes e^m)$ by 1

$\Longrightarrow$ a direct definition in terms of $M^{\mathcal{V}}(n)$

but I do not know it for $\mathcal{V}$.

<u>Prop.</u> $\varepsilon_1\varepsilon_2 \log Z^{in}(\varepsilon_1,\varepsilon_2,a,m,\Lambda)$ is regular at $\varepsilon_1=\varepsilon_2=0$

(1)

$$=: F_0^{in}(a,m,\Lambda) + (\varepsilon_1+\varepsilon_2)\times H^{in}(a,m,\Lambda)$$

$$+ \underbrace{\varepsilon_1\varepsilon_2}_{=\chi(\mathbb{C}^2)} A^{in}(a,m,\Lambda) + \underbrace{\frac{\varepsilon_1^2+\varepsilon_2^2}{3}}_{=\sigma(\mathbb{C}^2)} B^{in}(a,m,\Lambda) + \text{higher}$$

(2) $H^{in}=0$

Th. [again = the explicit formula in the previous Thm 1)]

$$\mathcal{D}^{\xi}(\exp(\alpha z + p x))$$

$$= \sum_{\mathfrak{s}} SW(\mathfrak{s}) \operatorname*{Res}_{a=\infty} \mathcal{B}(\mathfrak{s}, \xi ; a) da$$

where

$$\mathcal{B}(\mathfrak{s}, \xi ; a) da = \frac{da}{a} (-1)^{*} 2^{**} \qquad \xi' := c_1(\mathfrak{s}) - (\xi - K_X)$$

$$\left(\frac{2a}{\Lambda}\right)^{((\xi - K_X)^2) + (K_X^2) + 3\chi_h(X) - 2(\xi - K_X, c_1(\mathfrak{s}))} \exp\left(-(\xi - K_X - c_1(\mathfrak{s}), \alpha) a z - a^2 x\right)$$

$$\exp\Big[\frac{1}{3} \frac{\partial F_0^{in}}{\partial \log \Lambda} x + \left(\frac{1}{8}\frac{\partial^2 F_0^{in}}{\partial a^2} + \frac{1}{4}\frac{\partial^2 F_0^{in}}{\partial a \partial m} + \frac{1}{8}\frac{\partial^2 F_0^{in}}{\partial m^2}\right)((\xi - K_X)^2)$$

$$- \frac{1}{4}\left(\frac{\partial^2 F_0^{in}}{\partial a \partial m} + \frac{\partial^2 F_0^{in}}{\partial a^2}\right)(\xi - K_X, c_1(\mathfrak{s}))$$

$$+ \frac{1}{6}\left(\frac{\partial^2 F_0^{in}}{\partial a \partial \log \Lambda} + \frac{\partial^2 F_0^{in}}{\partial m \partial \log \Lambda}\right)(\xi - K_X, \alpha) z - \frac{1}{6}\frac{\partial^2 F_0^{in}}{\partial a \partial \log \Lambda}(c_1(\mathfrak{s}), \alpha) z$$

$$+ \frac{1}{18}\frac{\partial^2 F_0^{in}}{\partial \log \Lambda^2}(\alpha^2) z^2 + \chi_h(X)(12 A^{in} - 8 B^{in})$$

$$+ (K_X^2)\left(B^{in} - A^{in} + \frac{1}{8}\frac{\partial^2 F_0^{in}}{\partial a^2}\right)\Big]$$

evaluated at $(a, m = a, \Lambda^{4/3} a^{-1/3})$

Conjecture

This is also true for C^∞-4 mfd X,
where we understand K_X as a choice of a spin^c structure.

Remark In Feehan-Leness approach, need to choose a spin^c structure, in order to consider SO(3)-monopole equation